

Duality

About two dozen centuries ago (circa 400 BCE) a young man known to us as Theaetetus expanded the acknowledged collection of “perfectly symmetrical” solid polyhedra from three to five. When Theaetetus entered Plato’s Lyceum, the tetrahedron, the cube, and the dodecahedron, were all well known. Still, it was not known, and perhaps it had not been asked, if there were other solids with such perfect symmetries.

Theaetetus did ask, and upon asking he discovered the octahedron and the icosahedron. He did this by a simple trick which today we would call “duality,” though the word duality is used in far too many ways in mathematics.

First, recall the simplest features of the symmetric solids that were known before Theaetetus.

Solid	Faces	Vertices
Tetrahedron	4	4
Cube	6	8
Dodecahedron	12	20

Now, we can imagine Theaetetus asking himself how to construct a *new* symmetric object *by building on a symmetric object that he already knows*. So stated, it does take long to hit upon the idea of putting a vertex in the center of each face of the known object. These face-centered vertices now form a new polyhedron that inherits perfect symmetry. We just need to work out its properties.

In the case of the tetrahedron, we may be disappointed. The construction just gives us back a tetrahedron. Luckily, the cube gives us something new, simple, and lovely. It’s the octahedron (eight faces, six vertices). Also, the dual of the dodecahedron is a new solid, and it is surely the most beautiful of Theaetetus’s discoveries. It’s the icosahedron (twenty faces, twelve vertices).

Solid	Faces	Vertices	Dual	Faces	Vertices
Tetrahedron	4	4	Tetrahedron	4	4
Cube	6	8	Octahedron	8	6
Dodecahedron	12	20	Icosahedron	20	12

What favorite object do you know that could have a nice “dual”? Here, of course, “dual” just means “something constructed by a simple, symmetrical, transformation starting with something you like.” Naturally, such a transformation is sweetest when — as in Theaetetus’s cases — the dual of the dual is the original. Still, you should not overly constrain your search. A “dual” can be wonderful without being perfect!

J. Michael Steele

<http://www-stat.wharton.upenn.edu/~steele/>

KEYWORDS: geometric duality, invention, symmetry, construction